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Invariant Randers metrics on homogeneous Riemannian manifolds*

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Abstract

This paper studies Randers metrics on homogeneous Riemannian manifolds. It turns out that we can give a complete description of the invariant Randers metrics on a homogeneous Riemannian manifold as well as the geodesics, the flag curvatures. This result provides a convenient method to construct globally defined Berwald space which is neither Riemannian nor locally Minkowskian and gives another explanation of the example of Bao *et al* (1999 *An Introduction to Riemannian–Finsler Geometry* (Berlin: Springer)).

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Introduction

Randers spaces were first introduced by Randers in 1941 [2], when he studied the metric problems in the 4-space of general relativity. They also occur naturally in many other physical applications, especially in electron optics. According to Ingarden's account [3], the Lagrangian of relativistic electrons gives rise to a Randers metric which involves the normalized versions of the electric and magnetic potentials and the normalization involves the physical constants of the theory. In geometry, they also provide a rich source of explicit examples of y-global Berwald spaces, particularly those which are neither Riemannian nor locally Minkowskian. Since they are most closely related to Riemannian metrics among the class of Finsler spaces, many new geometric invariants are first computed for them (cf, for example, Bao and Lackey [4]).

The construction of Randers space is not an easy task. By a theorem from the Japanese school, in order to construct a Randers space of Berwald type which is neither Riemannian nor Minkowskian, one must find a nonzero globally defined parallel 1-form on a Riemannian manifold (cf [1]). This in general contains a tedious computation.

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This paper provides a convenient method to construct Randers metrics on homogeneous Riemannian manifold. Let G/H be a homogeneous Riemannian manifold. Then we prove that there exists a bijection between the set of all invariant Randers metrics with the underlying Riemannian metric of G/H and the open ball of radius 1 of the subspace of invariants of the adjoint action of H on the tangent space $T_o(G/H)$ at the original of G/H. In general, the corresponding Randers space is of Berwald type which is neither Riemannian nor locally Minkowskian. In the mean time, we describe the geodesics and obtain a formula for the computation of flag curvatures of these metrics. Our result provides many new interesting examples of Finsler space of Berwald type and gives another explanation of the example of [1].

1. Global expression of Randers metric

1.1. Let M be a smooth n-dimensional manifold. A Randers metric on M consists of a Riemannian metric $\tilde{a} := \tilde{a}_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j$ on M and a 1-form $\tilde{b} := \tilde{b}_i\,\mathrm{d}x^i$. By $\tilde{a}$ and $\tilde{b}$ we define a function F on TM

$$F(x, y) = \alpha(x, y) + \beta(x, y) \qquad x \in M \qquad y \in T_x(M)$$

where $\alpha(x, y) = \sqrt{\tilde{a}_{ij} y^i y^j}$, $\beta(x, y) = \tilde{b}_i(x) y^i$. F is a Finsler structure if and only if

$$\|\tilde{b}\| := \sqrt{\tilde{b}_i \tilde{b}^i} < 1 \tag{1.1}$$

where $\tilde{b}^i = \tilde{a}^{ij}\tilde{b}_j$, and $(\tilde{a}^{ij})$ is the inverse of the matrix $(\tilde{a}_{ij})$.

Here we introduce a global way to express a Randers metric on a Riemannian manifold, which is convenient when we consider such structures on homogeneous Riemannian manifolds. Let $x \in M$. Then the Riemannian metric induces an inner product in the cotangent space $T_x^*(M)$ in a standard way. An easy computation shows that $\langle \mathrm{d}x_i, \mathrm{d}x_j \rangle = \tilde{a}^{ij}(x)$. This inner product defines a linear isomorphism between $T_x^*(M)$ and $T_x(M)$. In this way the 1-form $\tilde{b}$ corresponds to a smooth vector field $\tilde{b}^\#$ on M. Let

$$\tilde{b}^\# = (\tilde{b}^\#)^i \partial/\partial x_i.$$

Then we have

$$(\tilde{b}^\#)^i = \sum_{j=1}^{n} \tilde{a}^{ij} \tilde{b}_j = \tilde{b}^i.$$

For any $y \in T_x(M)$ we have

$$\langle y, \tilde{b}^\# \rangle = \left\langle y, \left(\sum_{j=1}^{n} \tilde{a}^{ij}(x) \tilde{b}_j \right) \partial/\partial x_i \right\rangle = \tilde{b}_i(x) y^i = \beta(x, y).$$

It is obvious that $\|\tilde{b}\| = \|\tilde{b}^\#\|$. Thus (1.1) holds if and only if $\|\tilde{b}^\#\| < 1$.

In summarizing,

Lemma 1.1. *The Randers metric on a manifold consisting of a Riemannian metric*

$$\tilde{a} := \tilde{a}_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j$$

together with a smooth vector field $\tilde{b}^\#$ *with* $\tilde{a}(x)(\tilde{b}^\#, \tilde{b}^\#) < 1, \forall x \in M$, *is defined by*

$$F(x, y) = \sqrt{\tilde{a}(x)(y, y)} + \tilde{a}(x)(\tilde{b}^\#, y) \qquad x \in M \quad y \in T_x(M).$$

1.2. A theorem from the Japanese school asserts that the Randers metric defined by Riemannian metric $\tilde{a}$ and 1-form $\tilde{b}$ is of Berwald type if and only if $\tilde{b}$ is parallel with respect to $\tilde{a}$ (cf [1]).

It is obvious that $\tilde{b}$ is parallel if and only if the corresponding vector field $\tilde{b}^{\#}$ is parallel with respect to $\tilde{a}$.

Lemma 1.2. *Let F be a Randers metric on M defined by the Riemannian metric $\tilde{a}$ and the vector field $\tilde{b}^{\#}$. Then (M, F) is of Berwald type if and only if $\tilde{b}^{\#}$ is parallel with respect to $\tilde{a}$.*

1.3. Let us consider the group of isometries of a Randers space (M, F). Denote this group by $I(M, F)$. Let (M, F_1) be an arbitrary Finsler space and d be the distance function of (M, F_1) (cf [1]). Let ϕ be a one-to-one mapping of M onto itself which preserves d. In [5] we have proved that ϕ is differentiable and hence a differmorphism. Therefore, it is reasonable to define the group of isometries of (M, F_1) as the set of differmorphisms of M such that $F_1(x, y) = F_1(\phi(x), \mathrm{d}\phi_x(y)), \forall x \in M, y \in T_x(M)$. Denote this group by $I(M, F_1)$. It was proved in [5] that $I(M, F_1)$ is a Lie transformation group on M.

Proposition 1.3. *Let (M, F) be a Randers space with F defined by the Riemannian metric $\tilde{a}$ and the vector field $\tilde{b}^{\#}$. Then the group of isometries of (M, F) is a closed subgroup of the group of isometries of the Riemannian manifold $(M, \tilde{a})$.*

Proof. Let ϕ be an isometry of (M, F). Let $p \in M$ and denote $q = \phi(p)$. For any $y \in T_p(M)$ we have

$$\begin{aligned} F(p, y) &= \sqrt{\tilde{a}(p)(y, y)} + \tilde{a}(p)(\tilde{b}^{\#}, y) = F(q, \mathrm{d}\phi_p(y)) \\ &= \sqrt{\tilde{a}(q)(\mathrm{d}\phi_p(y), \mathrm{d}\phi_p(y))} + \tilde{a}(q)(b^{\#}, \mathrm{d}\phi_p(y)). \end{aligned} \tag{1.2}$$

Substituting y with $-y$ in (1.2) we obtain

$$\sqrt{\tilde{a}(p)(y, y)} - \tilde{a}(p)(\tilde{b}^{\#}, y) = \sqrt{\tilde{a}(q)(\mathrm{d}\phi_p(y), \mathrm{d}\phi_p(y))} - \tilde{a}(q)(b^{\#}, \mathrm{d}\phi_p(y)). \tag{1.3}$$

Taking the summation of (1.2) and (1.3) we get

$$\tilde{a}(p)(y, y) = \tilde{a}(\mathrm{d}\phi_p(y), \mathrm{d}\phi_p(y)) \qquad \tilde{a}(p)(\tilde{b}^{\#}, y) = \tilde{a}(q)(\tilde{b}^{\#}, \mathrm{d}\phi_p(y)).$$

Thus ϕ is an isometry with respect to the underlying Riemannian metric $\tilde{a}$ and for any $p \in M$, $\mathrm{d}\phi_p(\tilde{b}^{\#}|_p) = \tilde{b}^{\#}|_{\phi(p)}$. Therefore $I(M, F)$ is a closed subgroup of $I(M, \tilde{a})$. □

2. Invariant Randers metric on homogeneous manifolds

2.1. Let G/H be a reductive homogeneous manifold (cf Nomizu [6]). Let $\mathfrak{g} = \mathrm{Lie}\, G$, $\mathfrak{h} = \mathrm{Lie}\, H$. Fix a decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} \quad \text{(direct sum ofsubspace)} \tag{2.1}$$

where $\mathfrak{m}$ is a subspace of $\mathfrak{g}$ and satisfies

$$\mathrm{Ad}(h)\mathfrak{m} \subset \mathfrak{m} \qquad \forall\, h \in H.$$

From the results of section 1, we see that to construct invariant Randers metric on G/H, we first need to find invariant vector fields on G/H. The following proposition gives a complete description for invariant vector fields.

Proposition 2.1. *There exists a bijection between the set of invariant vector fields on G/H and the subspace*

$$V = \{X \in \mathfrak{m} | \mathrm{Ad}(h)X = X, \forall h \in H\}.$$

Proof. Let $\pi : G \to G/H$ be the natural projection, L_g and R_g be the left and right translations of G by g, respectively. The mapping π has a differential $\mathrm{d}\pi$ which maps $\mathfrak{g}$ onto the tangent space $T_o(G/H)$ to G/H at the original $o = \{H\}$. The kernel of $\mathrm{d}\pi$ is $\mathfrak{h}$. The translation $\tau(g): xH \to gxH$ satisfies

$$\pi \circ L_g = \tau(g) \circ \pi$$

and since for $h \in H, \pi \circ R_h = \pi$, $\mathrm{Ad}(g)X = \mathrm{d}R_{g^{-1}} \circ \mathrm{d}L_g(X)$. We have for the differentials

$$\mathrm{d}\pi \circ \mathrm{Ad}(h)X = \mathrm{d}\tau(h)_o \circ \mathrm{d}\pi(X) \qquad X \in \mathfrak{g}.$$

Thus under the isomorphism $\mathfrak{g}/\mathfrak{h} \simeq T_o(G/H)$ the linear transformation $\mathrm{Ad}(h)$ of $\mathfrak{g}/\mathfrak{h}$ corresponds to the linear transformation $\mathrm{d}\tau(h)_o$ of $T_o(G/H)$.

Now the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ gives a natural isomorphism

$$\mathfrak{g}/\mathfrak{h} \simeq \mathfrak{m}.$$

Under this isomorphism the linear transformation $\mathrm{d}\tau(h)_o$ of $T_o(G/H)$ corresponds to the linear transformation $\mathrm{Ad}(h)$ of $\mathfrak{m}$.

Now given $X \in V$, let $\tilde{X}_o$ be its image under the isomorphism $\mathfrak{m} \simeq T_o(G/H)$. Let $g \in G$ define a tangent vector at gH by

$$\tilde{X}_{gH} = \mathrm{d}(\tau(g))_o(\tilde{X}_o).$$

If $g_1H = gH$, then $g^{-1}g_1 \in H$. Since $\mathrm{Ad}(h)X = X, \forall h \in H$, the above argument shows that $\mathrm{d}\tau(g^{-1}g_1)_o\tilde{X}_o = \tilde{X}_o$. Thus $\mathrm{d}\tau(g)_o\tilde{X}_o = \mathrm{d}\tau(g_1)_o\tilde{X}_o$. Therefore $\tilde{X}$ is a well-defined vector field on G/H and it is obviously invariant under the action of G. That the correspondence $X \to \tilde{X}$ is a bijection is easy to verify.

2.2. Let G/H be a reductive homogeneous manifold. By proposition 1.3, the underlying Riemannian metric of an invariant Randers metric on G/H must be invariant. Therefore we first fix an invariant Riemannian metric $\tilde{a}$ on G/H and then consider the invariant Randers metrics on G/H with the underlying Riemannian metric $\tilde{a}$.

The invariant Riemannian metric $\tilde{a}$ induces an inner product $\langle,\rangle$ on $\mathfrak{g}$ which satisfies

$$\langle \mathrm{Ad}(h)X, \mathrm{Ad}(h)Y\rangle = \langle X, Y\rangle \qquad X, Y \in \mathfrak{g} \quad h \in H. \tag{2.2}$$

The subspace $\mathfrak{m}$ in (2.1) can be taken to be the orthogonal complement of $\mathfrak{h}$ with respect to this inner product. □

Theorem 2.2. *Let $\tilde{a}$ be an invariant Riemannian metric on G/H. Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to the inner product induced on $\mathfrak{g}$ by $\tilde{a}$. Then there exists a bijection between the set of all invariant Randers metrics on G/H with the underlying Riemannian metric $\tilde{a}$ and the set*

$$V_1 = \{X \in \mathfrak{m} | \mathrm{Ad}(h)X = X, \langle X, X\rangle < 1, \forall h \in H\}.$$

And for any $X \in V_1$, the corresponding Randers metric is of Berwald type. Furthermore, if G/H is not flat and $0 \neq X \in V_1$, then the corresponding Randers metric is neither Riemannian nor locally Minkowskian.

Proof. Let $X \in V_1$. By proposition 2.1, X corresponds to an invariant vector field $\tilde{X}$ on G/H. Since $\tilde{X}$ is invariant under the action of G, we have

$$\tilde{a}(gH)(\tilde{X}, \tilde{X}) = \tilde{a}(H)(\tilde{X}, \tilde{X}) = \langle X, X\rangle < 1.$$

By lemma 1.1 we can define a Randers metric F_X on G/H by

$$F_X(gH, y) = \sqrt{\tilde{a}(gH)(y, y)} + \tilde{a}(gH)(\tilde{X}, y), y \in T_{gH}(G/H).$$

F_X is obviously invariant under the action of G. It is easily seen that the correspondence $X \to F_X$ is a bijection.

For any $X \in V_1$, G acts transitively on G/H as the isometries of the corresponding Randers metric. Therefore G/H with the corresponding Randers metric is of Berwald type. By theorem 11.5.1 of [1], the vector field $\tilde{X}$ is parallel with respect to $\tilde{a}$. (This can be proved directly using the formula for the Levi-Civita connection of $\tilde{a}$ (cf Nomizu [6]).) By proposition 11.6.1 of [1] (p 305), if $\tilde{a}$ is not flat and $X \neq 0$, then the Randers metric is neither Riemannian nor locally Minkowskian. □

3. Geodesics and flag curvatures

Let G/H be a homogeneous manifold with invariant Riemannian metric g and F be an invariant Randers metric defined by the $\mathrm{Ad}(H)$-invariant vector X, where $g(X, X) < 1$. In this section we give a description for geodesics and curvatures of F.

It is an important fact that the connection of F is the same as that of the corresponding Riemannian metric g (cf [1]). Therefore the geodesics of F is the same as that of g. According to Nomizu [6], the geodesics through the original $o = \{H\}$ are the curves

$$\gamma_Y : t \mapsto \exp tY \cdot o \qquad (Y \in \mathfrak{m}).$$

The following theorem gives the formula for the flag curvatures.

Theorem 3.1. *Let Y be a nonzero vector in $\mathfrak{m}$ and P be a plane in $\mathfrak{m}$ containing Y. Then the flag curvature of the flag (P, Y) in $T_o(G/H)$ is given by*

$$K(P, Y) = \frac{2\sqrt{g(Y, Y)}}{2\sqrt{g(Y, Y)} + g(X, Y)} K(P)$$

where $K(P)$ is the Riemannian curvature of P of the Riemannian metric g, and

$$K(P) = \frac{g([[Y, U]_{\mathfrak{h}}, Y], U)}{g(U, U)g(Y, Y) - g^2(U, Y)}$$

where U is any vector in P such that $\mathrm{span}(Y, U) = P$ *and $[Y, U]_{\mathfrak{h}}$ is the orthogonal projection of $[Y, U]$ to $\mathfrak{h}$.*

Proof. According to Nomizu [6], the curvature tensor of F (and g) is

$$R(U, V)W = -[[U, V]_{\mathfrak{h}}, W].$$

Therefore

$$K(P, Y) = \frac{g_Y([[Y, U]_{\mathfrak{h}}, Y], U)}{g_Y(Y, Y)g_Y(U, U) - g_Y^2(U, Y)}.$$

Now for $s, t \in \mathbb{R}$,

$$\begin{aligned} F^2(Y + sU + tV) &= g(Y + sU + tV, Y + sU + tV) + g^2(X, Y + sU + tV) \\ &\quad + 2\sqrt{g(Y + sU + tV, Y + sU + tV)}\, g(X, Y + sU + tV). \end{aligned}$$

By a direct computation we get

$$\begin{aligned} g_Y(U, V) &= g(U, V) + \frac{1}{2}\frac{1}{\sqrt{g(Y, Y)}} g(U, V)g(X, Y) \\ &= g(U, V)\left(1 + \frac{g(X, Y)}{2\sqrt{g(Y, Y)}}\right). \end{aligned}$$

Therefore the flag curvature is

$$K(P,Y) = \frac{g_Y([[Y,U]_{\mathfrak{h}},Y],U)}{g_Y(Y,Y)g_Y(U,U)-g_Y^2(U,Y)}$$
$$= \frac{\left(1+\frac{g(X,Y)}{2\sqrt{g(Y,Y)}}\right)g([[Y,U]_{\mathfrak{h}},Y],U)}{\left(1+\frac{g(X,Y)}{2\sqrt{g(Y,Y)}}\right)^2(g(Y,Y)g(U,U)-g^2(U,Y))}$$
$$= \frac{2\sqrt{g(Y,Y)}}{2\sqrt{g(Y,Y)}+g(X,Y)}\frac{g([[Y,U]_{\mathfrak{h}},Y],U)}{g(Y,Y)g(U,U)-g^2(U,Y)}.$$

Combining this with the formula in Nomizu [6] completes the proof. □

Corollary 3.2. *Let G be a compact connected Lie group and H be a closed subgroup of G, Lie $G = \mathfrak{g}$, Lie $H = \mathfrak{h}$. Let g be a G-invariant Riemannian metric on G/H and $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$. Suppose the set*

$$V_1 = \{X \in \mathfrak{m} | \mathrm{Ad}(h)X = X, \forall h \in H, 0 < g(X,X) < 1\}$$

is not empty. Then there exists invariant Randers metrics on G/H. Let F be the invariant Randers metric defined by $X \in V_1$. Then for $Y \in \mathfrak{m}$, $g(Y,Y) = 1$, and a plane P in $\mathfrak{m}$ containing Y. The flag curvature of F of the flag (P,Y) is given by

$$K(P,Y) = \frac{2}{2+g(X,Y)}\left(\frac{1}{4}g([U,Y],[U,Y]) + \frac{3}{4}g([U,Y]_{\mathfrak{h}},[U,Y]_{\mathfrak{h}})\right)$$

where U is any unit vector in P orthogonal to Y.

Proof. According to Helgason [7], the sectional curvature of $(G/H, g)$ at P is

$$K(P) = \tfrac{1}{4}g([U,Y],[U,Y]) + \tfrac{3}{4}g([U,Y]_{\mathfrak{h}},[U,Y]_{\mathfrak{h}}).$$

Therefore the corollary follows. □

4. Some examples

4.1. Let G be a real Lie group. Then G can be viewed as a reductive homogeneous manifold with $H = \{e\}$ and $\mathfrak{m} = \mathfrak{g}$. Fix a left-invariant Riemannian metric $\tilde{a}$ on G. $\tilde{a}$ induces an inner product $\langle , \rangle$ on $\mathfrak{g}$. The subspace V in proposition 2.1 is just $\mathfrak{g}$. Therefore the set V_1 in theorem 3.2 is

$$V_1 = \{X \in \mathfrak{g} | \langle X, X\rangle < 1\}.$$

For any $X \in V$ there corresponds a left-invariant Randers metric on G. If $\tilde{a}$ is not flat and $X \neq 0$, then the corresponding Randers space is neither Riemannian nor locally Minkowskian of Berwald type.

We can also consider the right-invariant and even bi-invariant Randers metric on Lie groups. We omit the details.

4.2. Let us consider symmetric spaces. Let G/K be a global symmetric space with G semisimple. In general, there may not exist an invariant Randers metric with the underlying Riemannian metric of G/K. In fact, using theorem 2.2 we can easily prove that if $n \geqslant 2$, $n \neq 3$, then there does not exist an invariant Randers metric on $S^n = SO(n+1)/SO(n)$. Note that S^3 is a compact Lie group, so there exists a bi-invariant Randers metric on it.

The situation will change a lot if we shift the subgroup K to a proper closed subgroup of it. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the decomposition of $\mathfrak{g}$ corresponding to the symmetric pair (G, K). Let $\mathfrak{p}_1$ be a proper subspace of $\mathfrak{p}$. The group K has an orthogonal representation on $\mathfrak{p}$. Let K_1 be the subgroup of K which leaves each point of $\mathfrak{p}_1$ fixed. The homogeneous manifold G/K_1 admits an invariant Riemannian metric such that G/K_1 is a submersion of the Riemannian manifold G/K. Since G/K is not flat, G/K_1 is not flat. By theorem 2.2, there exists an invariant Randers metric on G/K_1 which makes it into a Berwald space which is neither Riemannian nor locally Minkowskian.

Example 4.2. Consider $S^n = SO(n+1)/SO(n), n \geqslant 4$. We have $\mathfrak{g} = \mathfrak{so}(n+1)$, $\mathfrak{k} = \mathfrak{so}(n)$,

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & \alpha \\ -\alpha^t & 0 \end{pmatrix} \middle| \alpha \in \mathbb{R}^n \right\}. \tag{3.2}$$

For an integer q, $1 \leqslant q \leqslant n-1$, let $\mathfrak{p}_q$ be the subspace of $\mathfrak{p}$ with

$$\alpha = \begin{pmatrix} \alpha_{n-q} \\ 0 \end{pmatrix} \qquad \alpha_{n-q} \in \mathbb{R}^{n-q}$$

in (3.2). The subgroup of $SO(n)$ which leaves each point of $\mathfrak{p}_q$ fixed is $SO(q) \hookrightarrow SO(n)$ as

$$SO(q) = \left\{ \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix} \middle| A \in SO(q) \right\}.$$

Then by the above argument, there exists an invariant Randers metric on $M = SO(n+1)/SO(q)$ which is neither Riemannian nor locally Minkowskian. Note that

$$SO(n+1)/SO(q) = (SO(n+1)/SO(q) \times SO(p)) \times SO(p)$$

where $p = n+1-q$. Therefore M is a fibre bundle over the Grassmannian manifold $G_{p,q}(\mathbb{R})$ with fibres $SO(p)$.

4.3. Let us give another explanation of the example in the book of Bao, Chern and Shen ([1], p 306). The underlying manifold is $M = S^n \times S^1$, $n \geqslant 2$. The Riemannian metric $\tilde{a}$ is the product of the standard ones on S^n and S^1. Use the usual spherical coordinate on S^1, i.e., let t be such that $(\cos t, \sin t, 0)$ parametrizes S^1. Let $\tilde{b} = \epsilon \mathrm{d}t$, $0 < \epsilon < 1$. Then $\tilde{b}$ is globally defined on M and a Randers metric can be defined by $\tilde{a}$ and $\tilde{b}$ which makes M into a Berwald space which is neither Riemannian nor locally Minkowskian.

The manifold M with Riemannian metric $\tilde{a}$ is a homogeneous Riemannian manifold (actually it is a globally symmetric space). We can write

$$M = (SO(n+1)/SO(n)) \times S^1 = (SO(n+1) \times S^1)/(SO(n) \times \{e\}) = G/H$$

where e is the unity element of S^1. The Lie algebra of G is $\mathfrak{g} = \mathfrak{so}(n+1) + \mathbb{R}^1$ (direct sum of ideals) and Lie $H = \mathfrak{h} = \mathfrak{so}(n)$. We can take

$$\mathfrak{m} = \mathfrak{m}_1 + \mathbb{R}^1$$

where

$$\mathfrak{m}_1 = \left\{ \begin{pmatrix} 0 & \alpha \\ -\alpha^t & 0 \end{pmatrix} \middle| \alpha \in \mathbb{R}^n \right\}.$$

The action of H on $\mathfrak{m}$ is

$$\mathrm{Ad}(h)(A, t) = (hAh^{-1}, t) \qquad A \in \mathfrak{m}_1 \quad t \in \mathbb{R}^1 \quad h \in H.$$

Therefore the subspace V in proposition 2.1 is

$$V = \{(0, t) | t \in \mathbb{R}^1\}$$

and the set $V_1 = \{(0, \epsilon)|, \epsilon < 1\}$. Therefore by theorem 2.2, for each $0 < \epsilon < 1$ we can construct a globally defined invariant Randers metric of Berwald type on M which is neither Riemannian nor Minkowskian.

5. Conclusions

From the results of sections 1–4, we see that it is generally not difficult to find invariant Randers metrics on a homogeneous Riemannian manifold G/H. In fact, let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{g} =$ Lie G, $\mathfrak{h} =$ Lie H and $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to the Riemannian metric on G/H. Then theorem 2.2 asserts that there exist invariant Randers metrics on G/H if and only if the adjoint representation of H on $\mathfrak{m}$ has nonzero fixed points. This condition is generally easy to satisfy. Since Randers space has many physical applications, it is hopeful that the results of this paper will be useful in the study of some physical problems. The formulae of geodesics and flag curvatures in section 3 are essential in geometry because there are very few cases for which we can give an explicit description. The method of this paper may also be useful when we study special Finsler spaces (e.g., those with constant flag curvatures, cf [1]).

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